2.11 The Laplacian

We have now met two scalar functions related to the electric field, the potential function ϕ (see Eq. (2.16)) and the divergence, div $\mathbf{E}$. In Cartesian coordinates the relationships are expressed as follows:

$$\mathbf{E} = -\text{grad}\,\phi = -\left(\hat{\mathbf{x}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\phi}{\partial z}\right), \tag{2.66}$$

$$\text{div}\,\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}. \tag{2.67}$$

Equation (2.66) shows that the x component of $\mathbf{E}$ is $E_x = -\partial\phi/\partial x$. Substituting this and the corresponding expressions for E_y and E_z into Eq. (2.67), we get a relation between div $\mathbf{E}$ and ϕ:

$$\text{div}\,\mathbf{E} = -\text{div}\,\text{grad}\,\phi = -\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right). \tag{2.68}$$

The operation on ϕ that is indicated by Eq. (2.68), except for the minus sign, we could call "div grad," or "taking the divergence of the gradient of . . . ." The symbol used to represent this operation is ∇^2, called *the Laplacian operator*, or just *the Laplacian.* The expression

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.69}$$

is the prescription for the Laplacian in Cartesian coordinates. So we have

$$\boxed{\text{div}\,\mathbf{E} = -\nabla^2\phi} \tag{2.70}$$

The notation ∇^2 is explained as follows. With the vector operator ∇ given in Eq. (2.60), its square equals

$$\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{2.71}$$

the same as the Laplacian in Cartesian coordinates. So the Laplacian is often called "del squared," and we say "del squared ϕ," meaning "div grad ϕ." *Warning:* In other coordinate systems, spherical coordinates, for instance, the explicit forms of the gradient operator and the Laplacian operator are not so simply related. This is evident in the list of formulas at the beginning of Appendix F. It is well to remember that the fundamental definition of the Laplacian operation is "divergence of the gradient of."

We can now express directly a *local* relation between the charge density at some point and the potential function in that immediate neighborhood. Combining Eq. (2.70) with Gauss's law in differential form, div $\mathbf{E} = \rho/\epsilon_0$, we have

$$\boxed{\nabla^2\phi = -\frac{\rho}{\epsilon_0}} \tag{2.72}$$

Equation (2.72), sometimes called *Poisson's equation*, relates the charge density to the second derivatives of the potential. Written out in Cartesian coordinates it is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -\frac{\rho}{\epsilon_0}. \tag{2.73}$$

One may regard this as the differential expression of the relationship expressed by the integral in Eq. (2.18), which tells us how to find the potential at a point by summing the contributions of all sources near and far.[4]

Example (Poisson's equation for a sphere) Let's verify that Eq. (2.72) holds for the potential due to a sphere with radius R and uniform charge density ρ. This potential was derived in the second example in Section 2.2. Spherical coordinates are the best choice here, so we will invoke the expression for the Laplacian in spherical coordinates, given in Eq. (F.3) in Appendix F. Since the potential depends only on r, we have $\nabla^2\phi = (1/r^2)\partial(r^2\,\partial\phi/\partial r)/\partial r$.

The potential outside the sphere is $\phi = \rho R^3/3\epsilon_0 r$. All that matters here is the fact that ϕ is proportional to $1/r$, because this makes $\partial\phi/\partial r$ proportional to $1/r^2$, from which we immediately see that $\nabla^2\phi = 0$. This agrees with Eq. (2.72), because $\rho = 0$ outside the sphere.

Inside the sphere, we have $\phi = \rho R^2/2\epsilon_0 - \rho r^2/6\epsilon_0$. The constant term vanishes when we take the derivative, so we have

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{-\rho r}{3\epsilon_0}\right) = -\frac{1}{r^2}\frac{\rho r^2}{\epsilon_0} = -\frac{\rho}{\epsilon_0}, \tag{2.74}$$

as desired.

2.12 Laplace's equation

Wherever $\rho = 0$, that is, in all parts of space containing no electric charge, the electric potential ϕ has to satisfy the equation

$$\nabla^2\phi = 0. \tag{2.75}$$

This is called *Laplace's equation.* We run into it in many branches of physics. Indeed one might say that from a mathematical point of view the

[4] In fact, it can be shown that Eq. (2.73) is the *mathematical* equivalent of Eq. (2.18). This means, if you apply the Laplacian operator to the integral in Eq. (2.18), you will come out with $-\rho/\epsilon_0$. We shall not stop to show how this is done; you'll have to take our word for it or figure out how to do it in Problem 2.27.

theory of classical fields is mostly a study of the solutions of this equation. The class of functions that satisfy Laplace's equation are called *harmonic functions.* They have some remarkable properties, one of which is the following.

Theorem 2.1 *If $\phi(x, y, z)$ satisfies Laplace's equation, then the average value of ϕ over the surface of any sphere (not necessarily a small sphere) is equal to the value of ϕ at the center of the sphere.*

Proof We can easily prove that this must be true of the electric potential ϕ in regions containing no charge. (See Section F.5 in Appendix F for a more general proof.) Consider a point charge q and a spherical surface S over which a charge q' is uniformly distributed. Let the charge q be brought in from infinity to a distance R from the center of the charged sphere, as in Fig. 2.27. The electric field of the sphere being the same as if its total charge q' were concentrated at its center, the work required is $qq'/4\pi\epsilon_0 R$.

Now suppose, instead, that the point charge q was there first and the charged sphere was later brought in from infinity. The work required for that is the product of q' and the average over the surface S of the potential due to the point charge q. Now the work is surely the same in the second case, namely $qq'/4\pi\epsilon_0 R$, so the average over the sphere of the potential due to q must be $q/4\pi\epsilon_0 R$. That is indeed the potential at the center of the sphere due to the external point charge q. That proves the assertion for any single point charge outside the sphere. But the potential of many charges is just the sum of the potentials due to the individual charges, and the average of a sum is the sum of the averages. It follows that the assertion must be true for *any* system of sources lying wholly outside the sphere. □

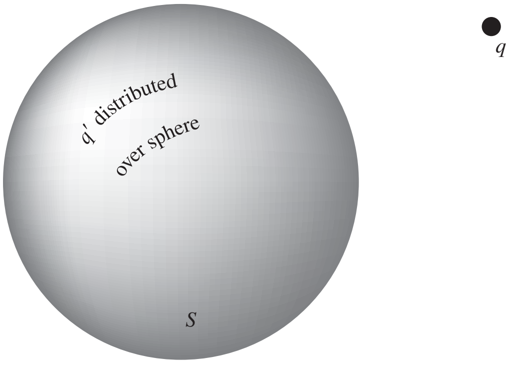


Figure 2.27.
The work required to bring in q' and distribute it over the sphere is q' times the *average*, over the sphere, of the potential ϕ due to q.

This property of the potential, that its average over an empty sphere is equal to its value at the center, is closely related to the following fact that you may find disappointing.

Theorem 2.2 *(Earnshaw's theorem) It is impossible to construct an electrostatic field that will hold a charged particle in stable equilibrium in empty space.*

This particular "impossibility theorem," like others in physics, is useful in saving fruitless speculation and effort. We can prove it in two closely related ways, first by looking at the field $\mathbf{E}$ and using Gauss's law, and second by looking at the potential ϕ and using the above fact concerning the average of ϕ over the surface of a sphere.

Proof First, suppose we have an electric field in which, contrary to the theorem, there *is* a point P at which a positively charged particle would be in stable equilibrium. That means that *any* small displacement of the particle from P must bring it to a place where an electric field acts to push it back toward P. But that means that a little sphere around P must have $\mathbf{E}$

pointing inward *everywhere* on its surface, which in turn means that there is a net inward flux through the sphere. This contradicts Gauss's law, for there is no negative source charge within the region. (Our charged test particle doesn't count; besides, it's positive.) In other words, you can't have an empty region where the electric field points all inward or all outward, and that's what you would need for *stable* equilibrium. Note that since this proof involved only Gauss's law, we could have presented this theorem back in Chapter 1.

A second proof, using Theorem 2.1, proceeds as follows. A stable position for a charged particle must be one where the potential ϕ is either lower than that at all neighboring points (if the particle is positively charged) or higher than that at all neighboring points (if the particle is negatively charged). Clearly neither is possible for a function whose average value over a sphere is always equal to its value at the center. □

Of course, one can have a charged particle in *equilibrium* in an electrostatic field, in the sense that the force on it is zero. The point where $\mathbf{E} = 0$ in Fig. 1.10 is such a location. The position midway between two equal positive charges is an equilibrium position for a third charge, either positive or negative. But the equilibrium is not stable. (Think what happens when the third charge is slightly displaced, either transversely or longitudinally, from its equilibrium position.) It *is* possible, by the way, to trap and hold stably an electrically charged particle by electric fields that vary in *time*. And it is certainly possible to hold stably a charged particle within a nonzero charge distribution. For example, a positive charge located at the center of a solid sphere of uniform negative charge is in stable equilibrium.

2.13 Distinguishing the physics from the mathematics

In the preceding sections we have been concerned with mathematical relations and new ways of expressing familiar facts. It may help to sort out physics from mathematics, and law from definition, if we try to imagine how things would be if the electric force were *not* a pure inverse-square force but instead a force with a finite range, for instance, a force varying like[5]

$$F(r) = \frac{e^{-\lambda r}}{r^2}. \tag{2.76}$$

Then Gauss's law in the integral form expressed in Eq. (2.50) would surely fail, for, by taking a very large surface enclosing some sources, we would find a vanishingly small field on this surface. The flux would go to zero as the surface expanded, rather than remain constant. However, we

[5] This force technically has an infinite range, but the exponential decay causes it to become essentially zero far away. So the range is finite, for all practical purposes.

could still define a field at every point in space. We could calculate the divergence of that field, and Eq. (2.51), which describes a mathematical property of *any* vector field, would still be true. Is there a contradiction here? No, because Eq. (2.52) would also fail. The divergence of the field would no longer be the same as the source density. We can understand this by noting that a small volume empty of sources could still have a net flux through it owing to the effect of a source *outside* the volume, if the field has finite range. As suggested in Fig. 2.28, more flux would enter the side near the source than would leave the volume.

Thus we may say that Eqs. (2.50) and (2.52) express the same *physical law*, the inverse-square law that Coulomb established by direct measurement of the forces between charged bodies, while Eq. (2.51) is an expression of a *mathematical theorem* that enables us to translate our statement of this law from differential to integral form or the reverse. The relations that connect **E**, ρ, and ϕ are gathered together in Fig. 2.29(a). The analogous expressions in Gaussian units are shown in Fig. 2.29(b).

How can we justify these differential relations between source and field in a world where electric charge is really not a smooth jelly but is concentrated on particles whose interior we know very little about? Actually, a statement like Eq. (2.72), Poisson's equation, is meaningful on a macroscopic scale only. The charge density ρ is to be interpreted as an average over some small but finite region containing many particles. Thus the function ρ cannot be continuous in the way a mathematician might prefer. When we let our region V_i shrink down in the course of demonstrating the differential form of Gauss's law, we know as physicists that we musn't let it shrink too far. That is awkward perhaps, but the fact is that we make out very well with the continuum model in large-scale

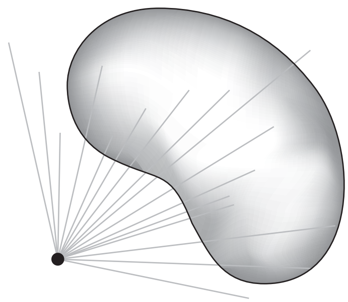


Figure 2.28.
In a non-inverse-square field, the flux through a closed surface is not zero.

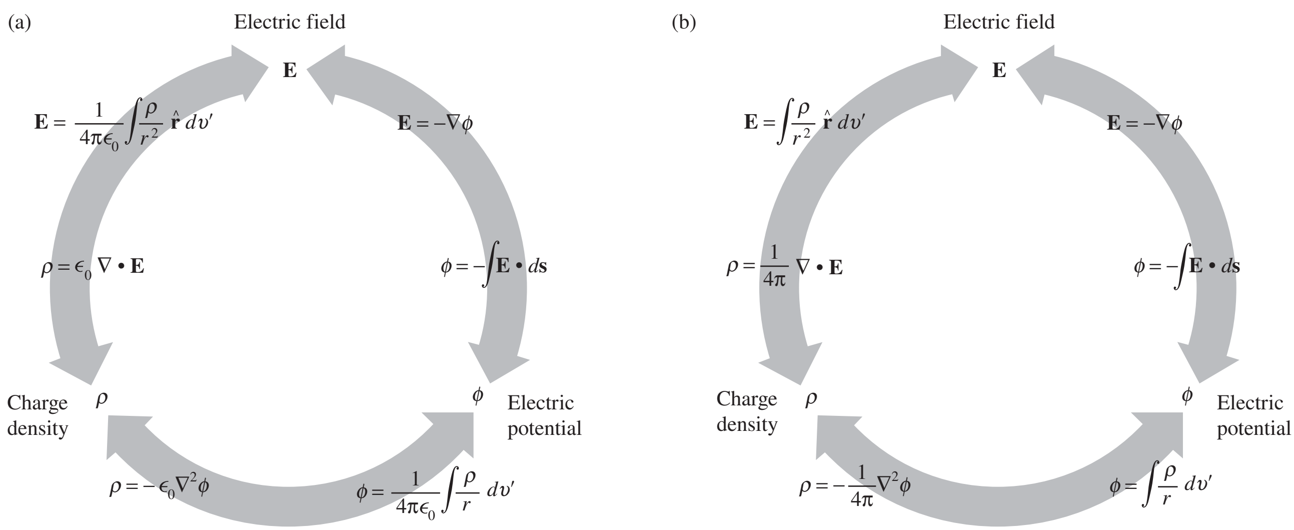


Figure 2.29.
(a) How electric charge density, electric potential, and electric field are related. The integral relations involve the line integral and the volume integral. The differential relations involve the gradient, the divergence, and div · grad (equivalently ∇^2), the Laplacian operator. The charge density ρ is in coulomb/meter^3, the potential ϕ is in volts, the field **E** is in volt/meter, and all lengths are in meters. (b) The same relations in Gaussian units. The charge density ρ is in esu/cm^3, the potential ϕ is in statvolts, the field **E** is in statvolt/meter, and all lengths are in centimeters.

electrical systems. In the atomic world we have the elementary particles, and vacuum. Inside the particles, even if Coulomb's law turns out to have some kind of meaning, much else is going on. The vacuum, so far as electrostatics is concerned, is ruled by Laplace's equation. Still, we cannot be sure that, even in the vacuum, passage to a limit of zero size has *physical* meaning.

2.14 The curl of a vector function

Note: Study of this section and the remainder of Chapter 2 can be postponed until Chapter 6 is reached. Until then our only application of the curl will be the demonstration that an electrostatic field is characterized by curl $\mathbf{E} = 0$*, as explained in Section 2.17. The reason we are introducing the curl now is that the derivation so closely parallels the above derivation of the divergence.*

We developed the concept of divergence, a local property of a vector field, by starting from the surface integral over a large closed surface. In the same spirit, let us consider the line integral of some vector field $\mathbf{F}(x, y, z)$, taken around a closed path, some curve C that comes back to join itself. The curve C can be visualized as the boundary of some surface S that spans it. A good name for the magnitude of such a closed-path line integral is *circulation*; we shall use Γ (capital gamma) as its symbol:

$$\Gamma = \int_C \mathbf{F} \cdot d\mathbf{s}. \tag{2.77}$$

In the integrand, $d\mathbf{s}$ is the element of path, an infinitesimal vector locally tangent to C (Fig. 2.30(a)). There are two senses in which C could be traversed; we have to pick one to make the direction of $d\mathbf{s}$ unambiguous. Incidentally, the curve C need not lie in a plane – it can be as crooked as you like.

Now bridge C with a new path B, thus making two loops, C_1 and C_2, each of which includes B as part of itself (Fig. 2.30(b)). Take the line integral around each of these, in the same directional sense. It is easy to see that the sum of the two circulations, Γ_1 and Γ_2, will be the same as the original circulation around C. The reason is that the bridge is traversed in opposite directions in the two integrations, leaving just the contributions that made up the original line integral around C. Further subdivision into many loops, $C_1, \ldots, C_i, \ldots, C_N$, leaves the sum unchanged:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{N} \int_{C_i} \mathbf{F} \cdot d\mathbf{s}_i, \qquad \text{or} \qquad \Gamma = \sum_{i=1}^{N} \Gamma_i. \tag{2.78}$$

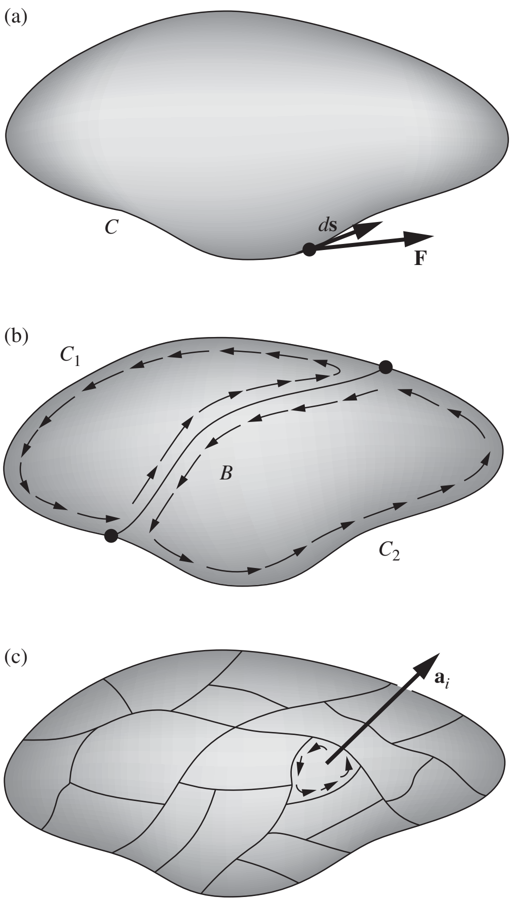


Figure 2.30.
For the subdivided loop, the sum of all the circulations Γ_i around the sections is equal to the circulation Γ around the original curve C.

In the same manner as in our discussion of divergence in Section 2.8, we can continue indefinitely to subdivide, now by adding new bridges instead of new surfaces, seeking in the limit to arrive at a quantity characteristic of the field **F** in a local neighborhood. When we subdivide the loops, we make loops with smaller circulation, but also with smaller area. So it is natural to consider the ratio of *loop circulation* to *loop area*, just as we considered in Section 2.8 the ratio of *flux* to *volume*. However, things are a little different here, because the area $\mathbf{a}_i$ of the bit of surface that spans a small loop C_i is really a vector (Fig. 2.30(c)), in contrast with the scalar volume V_i in Section 2.8. A surface has an orientation in space, whereas a volume does not. In fact, as we make smaller and smaller loops in some neighborhood, we can arrange to have a loop oriented in any direction we choose. (Remember, we are not committed to any particular surface over the whole curve C.) Thus we can pass to the limit in essentially different ways, and we must expect the result to reflect this.

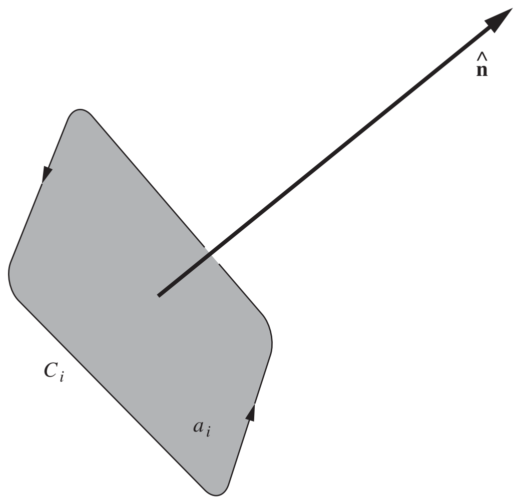


Figure 2.31.
Right-hand-screw relation between the surface normal and the direction in which the circulation line integral is taken.

Let us choose some particular orientation for the patch as it goes through the last stages of subdivision. The unit vector $\hat{\mathbf{n}}$ will denote the normal to the patch, which is to remain fixed in direction as the patch surrounding a particular point P shrinks down toward zero size. The limit of the ratio of *circulation* to *patch area* will be written this way:

$$\lim_{a_i \to 0} \frac{\Gamma_i}{a_i} \qquad \text{or} \qquad \lim_{a_i \to 0} \frac{\int_{C_i} \mathbf{F} \cdot d\mathbf{s}}{a_i}. \tag{2.79}$$

The rule for sign is that the direction of $\hat{\mathbf{n}}$ and the sense in which C_i is traversed in the line integral shall be related by a right-hand-screw rule, as in Fig. 2.31. The limit we obtain by this procedure is a scalar quantity that is associated with the point P in the vector field **F**, and with the direction $\hat{\mathbf{n}}$. We could pick three directions, such as $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, and get three different numbers. It turns out that these numbers can be considered components of a vector. We call the vector "curl **F**." That is to say, the number we get for the limit with $\hat{\mathbf{n}}$ in a particular direction is the component, in that direction, of the vector curl **F**. To state this in an equation,

$$(\text{curl}\,\mathbf{F}) \cdot \hat{\mathbf{n}} = \lim_{a_i \to 0} \frac{\int_{C_i} \mathbf{F} \cdot d\mathbf{s}}{a_i} \tag{2.80}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the curve C_i.

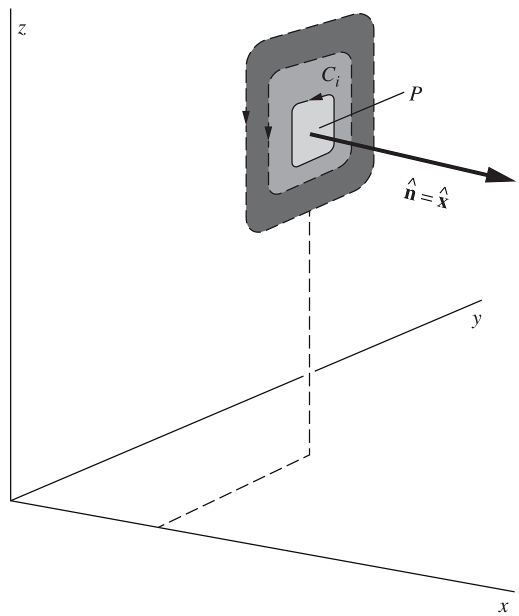


Figure 2.32.
The patch shrinks around P, keeping its normal pointing in the x direction.

For instance, the x component of curl **F** is obtained by choosing $\hat{\mathbf{n}} = \hat{\mathbf{x}}$, as in Fig. 2.32. As the loop shrinks down around the point P, we keep it in a plane perpendicular to the x axis. In general, the vector curl $\mathbf{F}$ will vary from place to place. If we let the patch shrink down around some other point, the ratio of circulation to area may have a different value, depending on the nature of the vector function $\mathbf{F}$. That is, curl $\mathbf{F}$ is itself a vector function of the coordinates. Its direction at each point in space is normal to the plane through this point in which the circulation is a maximum. Its magnitude is the limiting value of circulation per unit area, in this plane, around the point in question.

The last two sentences might be taken as a definition of curl $\mathbf{F}$. Like Eq. (2.80) they make no reference to a coordinate frame. We have not proved that the object so named and defined is a vector; we have only asserted it. Possession of direction and magnitude is not enough to make something a vector. The components as defined must behave like vector components. Suppose we have determined certain values for the x, y, and z components of curl $\mathbf{F}$ by applying Eq. (2.80) with $\hat{\mathbf{n}}$ chosen, successively, as $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. If curl $\mathbf{F}$ is a vector, it is uniquely determined by these three components. If some fourth direction is now chosen for $\hat{\mathbf{n}}$, the left side of Eq. (2.80) is fixed and the quantity on the right, the circulation in the plane perpendicular to the new $\hat{\mathbf{n}}$, had better agree with it! Indeed, until one is sure that curl $\mathbf{F}$ is a vector, it is not even obvious that there can be at most one direction for which the circulation per unit area at P is maximum – as was tacitly assumed in the latter definition. In fact, Eq. (2.80) does define a vector, but we shall not give a proof of that.

2.15 Stokes' theorem

From the circulation around an infinitesimal patch of surface we can now work back to the circulation around the original large loop C:

$$\Gamma = \int_C \mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{N} \Gamma_i = \sum_{i=1}^{N} a_i \left(\frac{\Gamma_i}{a_i} \right). \tag{2.81}$$

In the last step we merely multiplied and divided by a_i. Now observe what happens to the right-hand side as N is made enormous and all the a_i areas shrink. From Eq. (2.80), the quantity in parentheses becomes $(\text{curl}\,\mathbf{F}) \cdot \hat{\mathbf{n}}_i$, where $\hat{\mathbf{n}}_i$ is the unit vector normal to the ith patch. So we have on the right the sum, over all patches that make up the entire surface S spanning C, of the product "patch area times normal component of (curl $\mathbf{F}$)." This is simply the *surface integral*, over S, of the vector curl $\mathbf{F}$:

$$\sum_{i=1}^{N} a_i \left(\frac{\Gamma_i}{a_i} \right) = \sum_{i=1}^{N} a_i (\text{curl}\,\mathbf{F}) \cdot \hat{\mathbf{n}}_i \longrightarrow \int_S \text{curl}\,\mathbf{F} \cdot d\mathbf{a}, \tag{2.82}$$

because $d\mathbf{a} = a_i\hat{\mathbf{n}}_i$, by definition. We thus find that

$$\boxed{\int_C \mathbf{F}\cdot d\mathbf{s} = \int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{a}} \qquad \text{(Stokes' theorem)}. \tag{2.83}$$

The relation expressed by Eq. (2.83) is a mathematical theorem called *Stokes' theorem*. Note how it resembles Gauss's theorem, the divergence theorem, in structure. Stokes' theorem relates the line integral of a vector to the surface integral of the curl of the vector. Gauss's theorem, Eq. (2.49), relates the surface integral of a vector to the volume integral of the divergence of the vector. Stokes' theorem involves a surface and the curve that bounds it. Gauss's theorem involves a volume and the surface that encloses it.

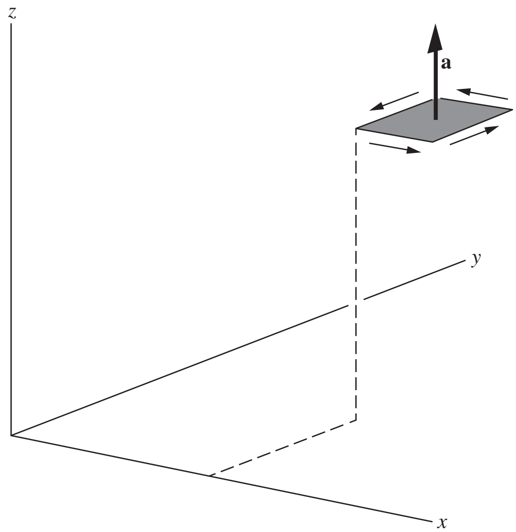


Figure 2.33.
Circulation around a rectangular patch with $\hat{\mathbf{n}} = \hat{\mathbf{z}}$.

2.16 The curl in Cartesian coordinates

Equation (2.80) is the fundamental definition of curl $\mathbf{F}$, stated without reference to any particular coordinate system. In this respect it is like our fundamental definition of divergence, Eq. (2.47). As in that case, we should like to know how to calculate curl $\mathbf{F}$ when the vector function $\mathbf{F}(x, y, z)$ is explicitly given. To find the rule, we carry out the integration called for in Eq. (2.80), but we do it over a path of very simple shape, one that encloses a rectangular patch of surface parallel to the xy plane (Fig. 2.33). That is, we are taking $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. In agreement with our rule about sign, the direction of integration around the rim must be clockwise as seen by someone looking up in the direction of $\hat{\mathbf{n}}$. In Fig. 2.34 we look down onto the rectangle from above.

The line integral of $\mathbf{A}$ around such a path depends on the variation of A_x with y and the variation of A_y with x. For if A_x had the same average value along the top of the frame, in Fig. 2.34, as along the bottom of the frame, the contribution of these two pieces of the whole line integral would obviously cancel. A similar remark applies to the side members. To the first order in the small quantities Δx and Δy, the difference between the average of A_x over the top segment of path at $y + \Delta y$ and its average over the bottom segment at y is

$$\left(\frac{\partial A_x}{\partial y}\right)\Delta y. \tag{2.84}$$

This follows from an argument similar to the one we used with Fig. 2.22(b):

$$A_x = A_x(x, y) + \frac{\Delta x}{2}\frac{\partial A_x}{\partial x} \qquad \left(\begin{array}{l}\text{at midpoint of}\\ \text{bottom of frame}\end{array}\right),$$

$$A_x = A_x(x, y) + \frac{\Delta x}{2}\frac{\partial A_x}{\partial x} + \Delta y\frac{\partial A_x}{\partial y} \qquad \left(\begin{array}{l}\text{at midpoint of}\\ \text{top of frame}\end{array}\right). \tag{2.85}$$

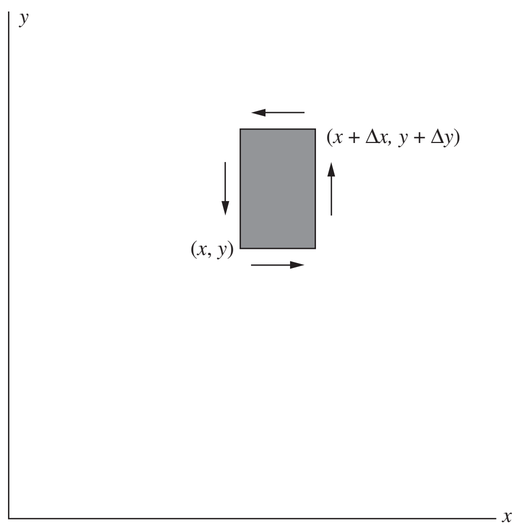


Figure 2.34.
Looking down on the patch in Fig. 2.33.

These are the average values referred to, to first order in the Taylor expansion. It is their difference, times the length of the path segment Δx, that determines their net contribution to the circulation. This contribution is $-\Delta x\,\Delta y\,(\partial A_x/\partial y)$. The minus sign comes in because we are integrating toward the left at the top, so that if A_x is more positive at the top, it results in a negative contribution to the circulation. The contribution from the sides is $\Delta y\,\Delta x\,(\partial A_y/\partial x)$, and here the sign is positive, because if A_y is more positive on the right, the result is a positive contribution to the circulation.

Thus, neglecting any higher powers of Δx and Δy, the line integral around the whole rectangle is

$$\int \mathbf{A}\cdot d\mathbf{s} = -\Delta x\cdot\left(\frac{\partial A_x}{\partial y}\right)\Delta y + \Delta y\cdot\left(\frac{\partial A_y}{\partial x}\right)\Delta x$$
$$= \Delta x\,\Delta y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right). \qquad (2.86)$$

Now $\Delta x\,\Delta y$ is the magnitude of the area of the enclosed rectangle, which we have represented by a vector in the z direction. Evidently the quantity

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \qquad (2.87)$$

is the limit of the ratio

$$\frac{\text{line integral around patch}}{\text{area of patch}} \qquad (2.88)$$

as the patch shrinks to zero size. If the rectangular frame had been oriented with its normal in the positive y direction, like the left frame in Fig. 2.35, we would have found the expression

$$\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \qquad (2.89)$$

for the limit of the corresponding ratio. And if the frame had been oriented with its normal in the positive x direction, like the right frame in Fig. 2.35, we would have obtained

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}. \qquad (2.90)$$

Although we have considered rectangles only, our result is actually independent of the shape of the little patch and its frame, for reasons much the same as in the case of the integrals involved in the divergence theorem. For instance, it is clear that we can freely join different

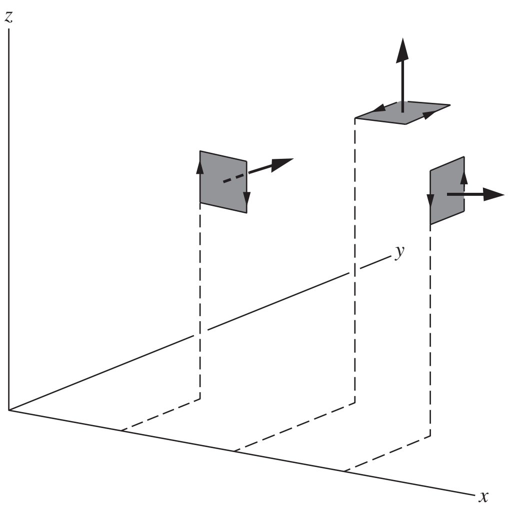


Figure 2.35.
For each orientation, the limit of the ratio circulation/area determines a component of curl $\mathbf{A}$ at that point. To determine all components of the vector curl $\mathbf{A}$ at any point, the patches should all cluster around that point; here they are separated for clarity.

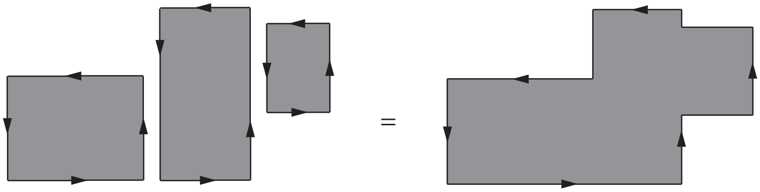


Figure 2.36.
The circulation in the loop on the right is the sum of the circulations in the rectangles, and the area on the right is the sum of the rectangular areas. This diagram shows why the circulation/area ratio is independent of shape.

rectangles to form other figures, because the line integrals along the merging sections of boundary cancel one another exactly (Fig. 2.36).

We conclude that, for any of these orientations, the limit of the ratio of circulation to area is independent of the shape of the patch we choose. Thus we obtain as a general formula for the components of the vector curl **F**, when **F** is given as a function of x, y, and z:

$$\text{curl}\,\mathbf{F} = \hat{\mathbf{x}}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right). \tag{2.91}$$

You may find the following rule easier to remember than the formula itself. Make up a determinant like this:

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}. \tag{2.92}$$

Expand it according to the rule for determinants, and you will get curl **F** as given by Eq. (2.91). Note that the x component of curl **F** depends on the rate of change of F_z in the y direction and the negative of the rate of change of F_y in the z direction, and so on.

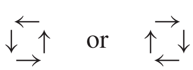
The symbol $\nabla\times$, read as "del cross," where ∇ is interpreted as the "vector"

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}, \tag{2.93}$$

is often used in place of the name *curl*. If we write $\nabla \times \mathbf{F}$ and follow the rules for forming the components of a vector cross product, we get automatically the vector curl **F**. So curl **F** and $\nabla\times\mathbf{F}$ mean the same thing.

2.17 The physical meaning of the curl

The name *curl* reminds us that a vector field with a nonzero curl has circulation, or vorticity. Maxwell used the name *rotation*, and in German a similar name is still used, abbreviated rot. Imagine a velocity vector field **G**, and suppose that curl **G** is not zero. Then the velocities in this field have something of this character:



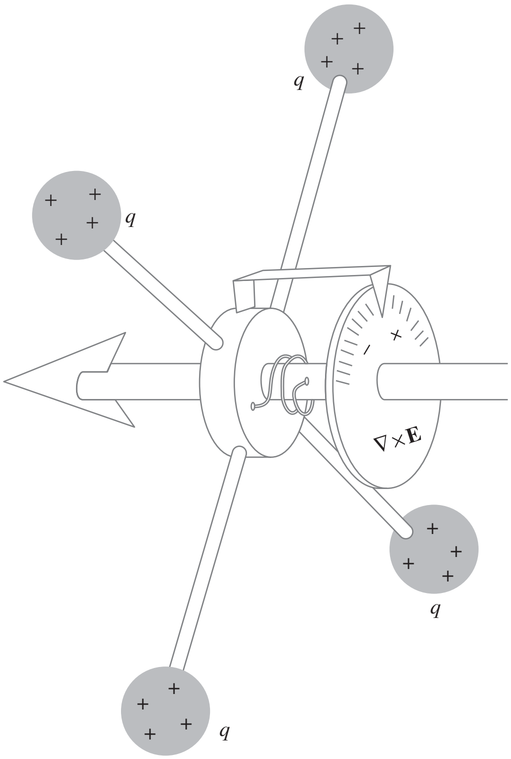


Figure 2.37.
The curlmeter.

superimposed, perhaps, on a general flow in one direction. For instance, the velocity field of water flowing out of a bathtub generally acquires a circulation. Its curl is not zero over most of the surface. Something floating on the surface rotates as it moves along. In the physics of fluid flow, hydrodynamics and aerodynamics, this concept is of central importance.

To make a "curlmeter" for an electric field – at least in our imagination – we could fasten positive charges to a hub by insulating spokes, as in Fig. 2.37. Exploring an electric field with this device, we would find, wherever curl **E** is not zero, a tendency for the wheel to turn around the shaft. With a spring to restrain rotation, the amount of twist could be used to indicate the torque, which would be proportional to the component of the vector curl **E** in the direction of the shaft. If we can find the direction of the shaft for which the torque is maximum and clockwise, that is the direction of the vector curl **E**. (Of course, we cannot trust the curlmeter in a field that varies greatly within the dimensions of the wheel itself.)

What can we say, in the light of all this, about the *electrostatic* field **E**? The conclusion we can draw is a simple one: the curlmeter will always read zero! That follows from a fact we have already learned; namely, in the electrostatic field the line integral of **E** around *any* closed path is zero. Just to recall why this is so, remember from Section 2.1 that the line integral of **E** between any two points such as P_1 and P_2 in Fig. 2.38 is independent of the path. (This then implies that **E** can be written as the negative gradient of the well-defined potential function given by Eq. (2.4).) As we bring the two points P_1 and P_2 close together, the line integral over the shorter path in the figure obviously vanishes – unless the final location is at a singularity such as a point charge, a case we can rule out. So the line integral must be zero over the closed loop in Fig. 2.38(d). But now, if the circulation is zero around *any* closed path, it follows from Stokes' theorem that the surface integral of curl **E** is zero over a patch of any size, shape, or location. But then curl **E** must be zero *everywhere*, for if it were not zero somewhere we could devise a patch in that neighborhood to violate the conclusion. We can sum all of this up by saying that if **E** equals the negative gradient of a potential function ϕ (which is the case for any electrostatic field **E**), then

$$\text{curl}\,\mathbf{E} = 0 \qquad \text{(everywhere).} \tag{2.94}$$

The converse is also true. If curl **E** is known to be zero everywhere, then **E** must be describable as the gradient of some potential function ϕ. This follows from the fact that zero curl implies that the line integral of **E** is path-independent (by reversing the above reasoning), which in turn implies that ϕ can be defined in an unambiguous manner as the negative line integral of the field. If curl $\mathbf{E} = 0$, then **E** could be an electrostatic field.

Example This test is easy to apply. When the vector function in Fig. 2.3 was first introduced, it was said to represent a possible electrostatic field. The components were specified by $E_x = Ky$ and $E_y = Kx$, to which we should add $E_z = 0$ to complete the description of a field in three-dimensional space. Calculating curl $\mathbf{E}$ we find

$$(\text{curl}\,\mathbf{E})_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0,$$

$$(\text{curl}\,\mathbf{E})_y = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0,$$

$$(\text{curl}\,\mathbf{E})_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = K - K = 0. \quad (2.95)$$

This tells us that $\mathbf{E}$ is the (negative) gradient of some scalar potential, which we know from Eq. (2.8), and which we verified in Eq. (2.17), is $\phi = -Kxy$. Incidentally, this particular field $\mathbf{E}$ happens to have zero divergence also:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0. \quad (2.96)$$

It therefore represents an electrostatic field in a *charge-free* region.

On the other hand, the equally simple vector function defined by $F_x = Ky$; $F_y = -Kx$; $F_z = 0$, does not have zero curl. Instead,

$$(\text{curl}\,\mathbf{F})_z = -2K. \quad (2.97)$$

Hence no electrostatic field could have this form. If you sketch roughly the form of this field, you will see at once that it has circulation.

Example (Field from a sphere) We can also verify that the electric field due to a sphere with radius R and uniform charge density ρ has zero curl. From the example in Section 1.11, the fields inside and outside the sphere are, respectively,

$$E_r^{\text{in}} = \frac{\rho r}{3\epsilon_0} \qquad \text{and} \qquad E_r^{\text{out}} = \frac{\rho R^3}{3\epsilon_0 r^2}. \quad (2.98)$$

As usual, we will work with spherical coordinates when dealing with a sphere. The expression for the curl in spherical coordinates, given in Eq. (F.3) in Appendix F, is unfortunately the most formidable one in the list. However, the above electric field has only a radial component, so only two of the six terms in the lengthy expression for the curl have a chance of being nonzero. Furthermore, the radial component depends only on r, being proportional to either r or $1/r^2$. So the two possibly nonzero terms, which involve the derivatives $\partial E_r/\partial\phi$ and $\partial E_r/\partial\theta$, are both zero ($\phi$ here is an angle, not the potential!). The curl is therefore zero. This result holds for *any* radial field that depends only on r. The particular r and $1/r^2$ forms of our field are irrelevant.

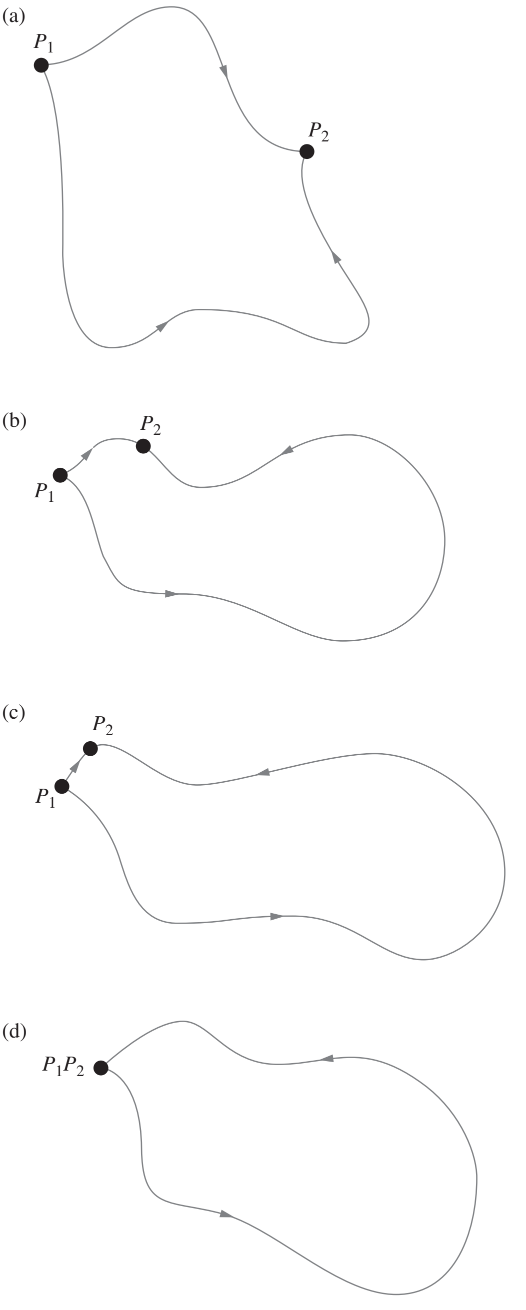


Figure 2.38.
If the line integral between P_1 and P_2 is independent of path, the line integral around a closed loop must be zero.

You can develop some feeling for these aspects of vector functions by studying the two-dimensional fields pictured in Fig. 2.39. In four of

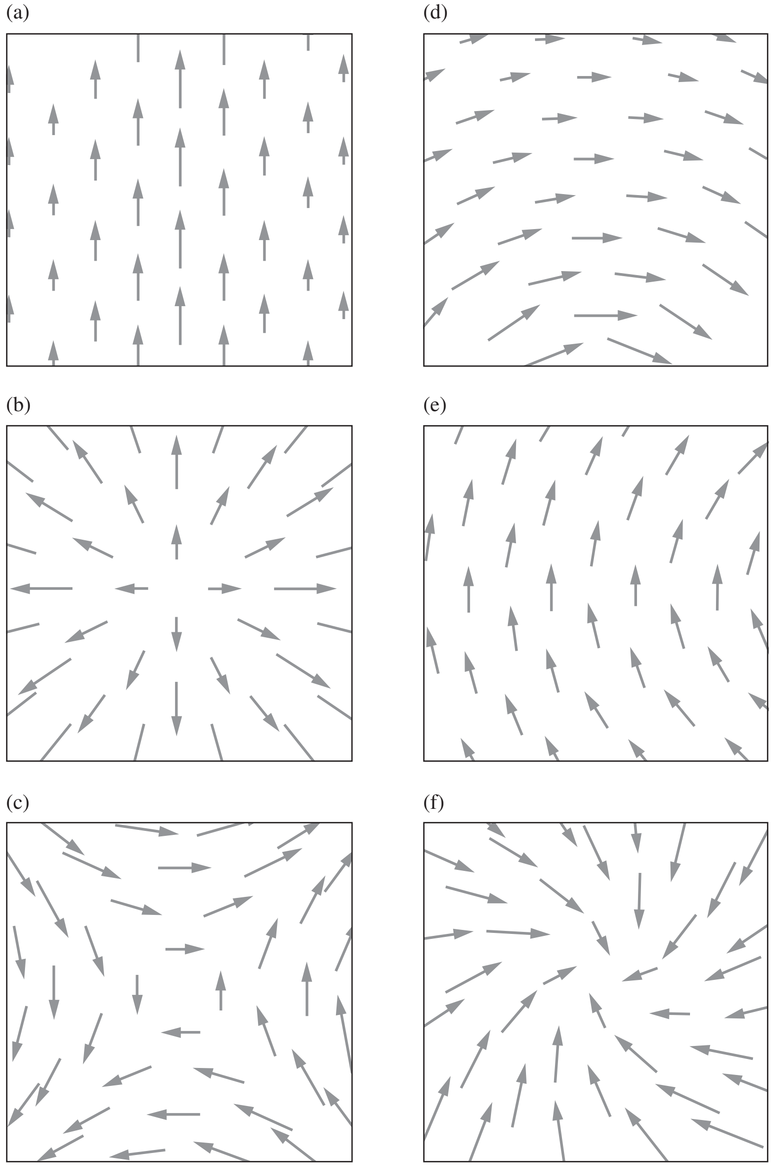


Figure 2.39.
Four of these vector fields have zero divergence in the region shown. Three have zero curl. Can you spot them?

these fields the divergence of the vector function is zero throughout the region shown. Try to identify the four. Divergence implies a net flux into, or out of, a neighborhood. It is easy to spot in certain patterns. In others you may be able to see at once that the divergence is zero. In three of the fields the curl of the vector function is zero throughout the region shown. Try to identify the three by deciding whether a line

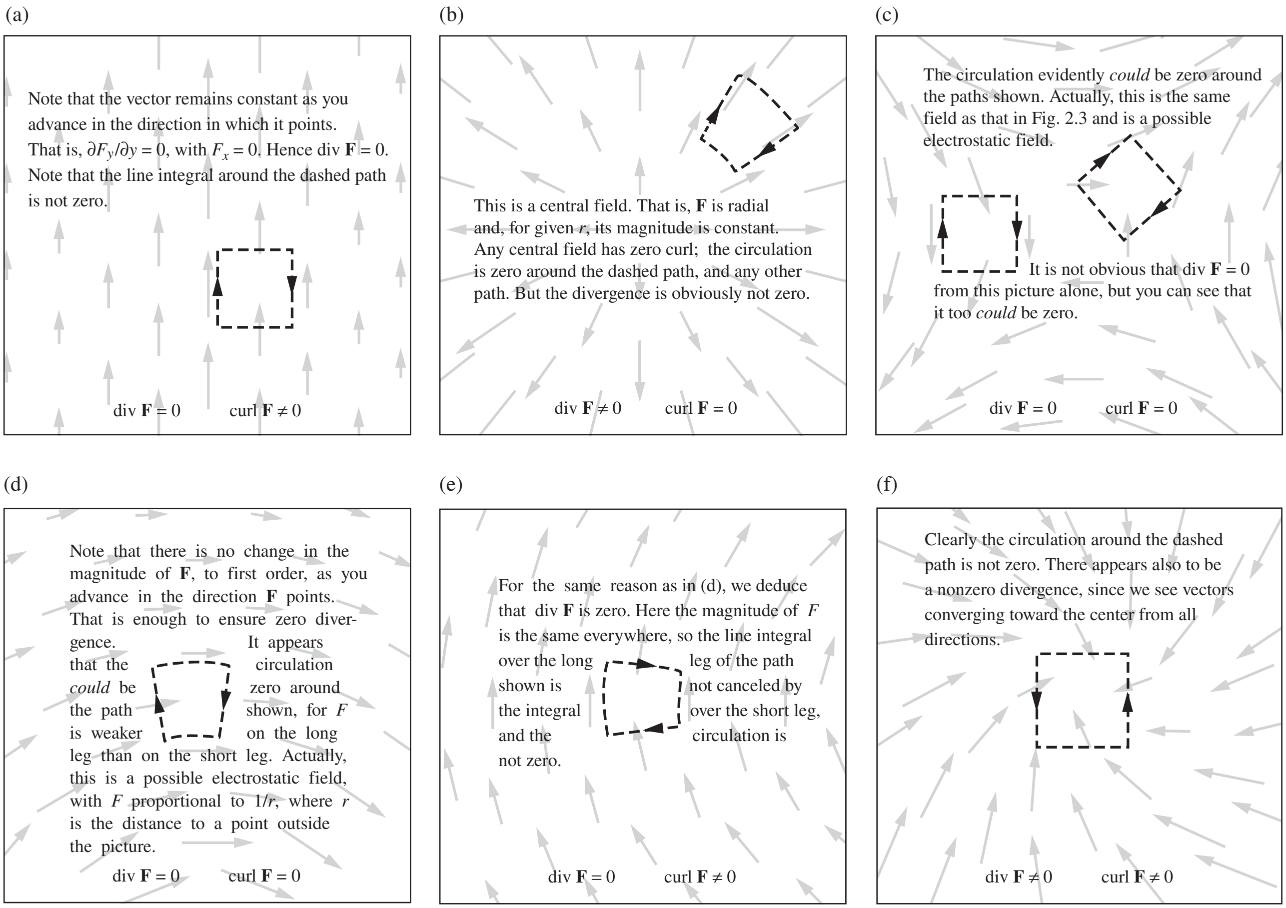
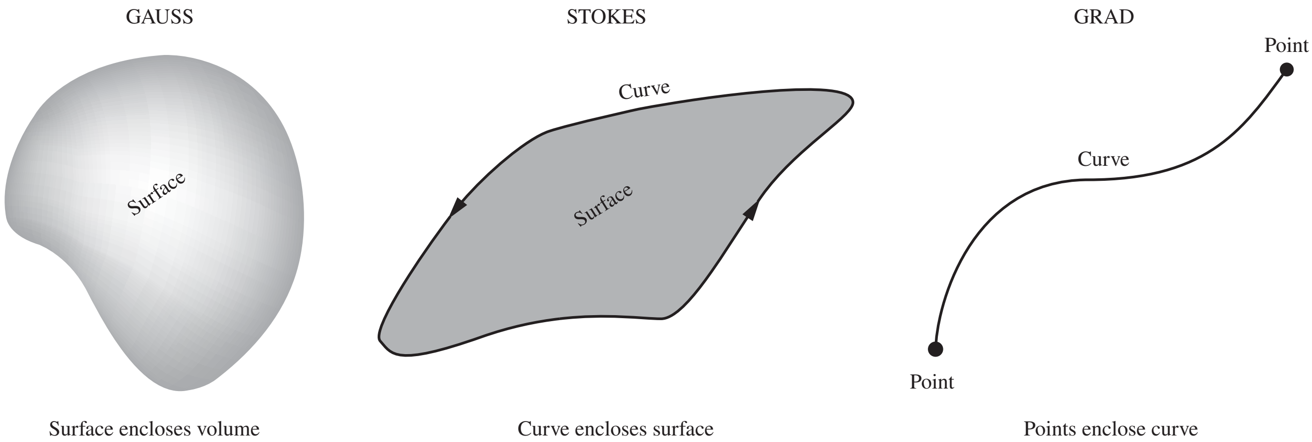


Figure 2.40.
Discussion of Fig. 2.39.

integral around any loop would or would not be zero in each picture. That is the essence of *curl*. After you have studied the pictures, think about these questions before you compare your reasoning and your conclusions with the explanation given in Fig. 2.40.

The curl of a vector field will prove to be a valuable tool later on when we deal with electric and magnetic fields whose curl is *not* zero. We have developed it at this point because the ideas involved are so close to those involved in the divergence. We may say that we have met two kinds of derivatives of a vector field. One kind, the divergence, involves the rate of change of a vector component in its own direction, $\partial F_x/\partial x$, and so on. The other kind, the curl, is a sort of "sideways derivative," involving the rate of change of F_x as we move in the y or z direction.



$$\int_{\text{surface}} \mathbf{F} \cdot d\mathbf{a} = \int_{\text{volume}} \operatorname{div} \mathbf{F}\, dv \qquad \int_{\text{curve}} \mathbf{A} \cdot d\mathbf{s} = \int_{\text{surface}} \operatorname{curl} \mathbf{A} \cdot d\mathbf{a} \qquad \phi_2 - \phi_1 = \int_{\text{curve}} \operatorname{grad} \phi \cdot d\mathbf{s}$$

In Cartesian coordinates, with $\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$:

$$\operatorname{div} \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \nabla \cdot \mathbf{F}$$

$$\operatorname{curl} \mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = \nabla \times \mathbf{A}$$

$$\operatorname{grad} \phi = \hat{\mathbf{x}}\frac{\partial \phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial \phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial \phi}{\partial z} = \nabla \phi$$

Figure 2.41.
Some vector relations summarized.

The relations called Gauss's theorem and Stokes' theorem are summarized in Fig. 2.41. The connection between the scalar potential function and the line integral of its gradient can also be looked on as a member of this family of theorems and is included in the third column. In all three of these theorems, the right-hand side of the equation involves an integral over an N-dimensional space, while the left-hand side involves an integral over the $(N-1)$-dimensional boundary of the space. In the "grad" theorem, this latter integral is simply the discrete sum over two points.

2.18 Applications

As mentioned in Section 1.16, the electrical breakdown of air occurs at a field of about $3 \cdot 10^6$ V/m. So if you shuffle your feet on a carpet and then